Mathematical Skills:
Functions

What is a Function?

- A mathematical function is a process that converts one set of numbers into another.
- For example: Doubling

Doubling Function	
Input	Output
1	2
2	4
3	6
4	8

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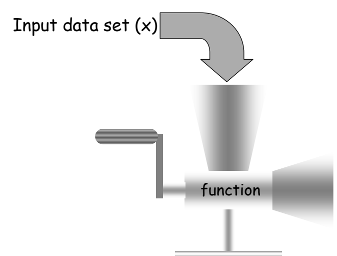
- For example: Doubling

Doubling Function	
Input (x)	Output (y)
1	2
2	4
3	6
4	8

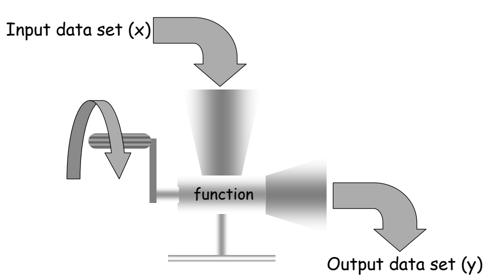
$y = 2 \times x$
or
$y = 2x$

- Important: For each input, there is only one possible output!

The Function Machine

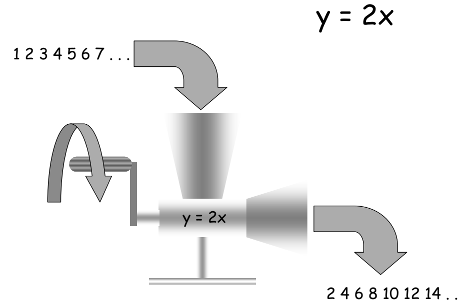


The Function Machine



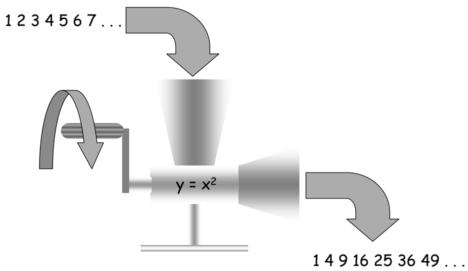
Examples

y = 2x



Examples

$y = x^2$



Function Notation

- Variables
  - Dependent Variable (*y*)
  - Independent Variable (*x*)
- Constants

$y = x + 2$
$f(x)= x + 2$

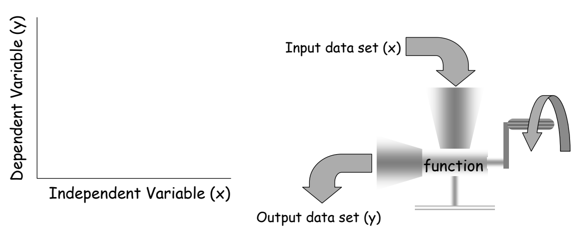
$y = x + z$
$f(x,z) = x + z$

Examples of Functions

- Circumference of a circle:
  - Circumference = 2 X π X radius
  - $f(r) = 2\pi r$
- Area of a circle:
  - Area = π X radius^2
  - $f(r) = \pi r^2$
- Volume of a sphere
  - Volume = 4/3 π X radius^3
  - $f(r) = 4/3\ \pi r^3$

Graphs of functions

- Functions generate pairs of numbers
  - These can be used as co-ordinates to draw graphs
  - Graphical display of function



Graphs of functions

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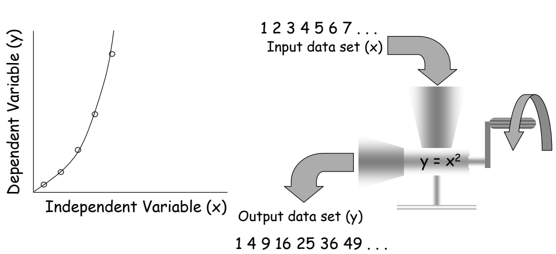
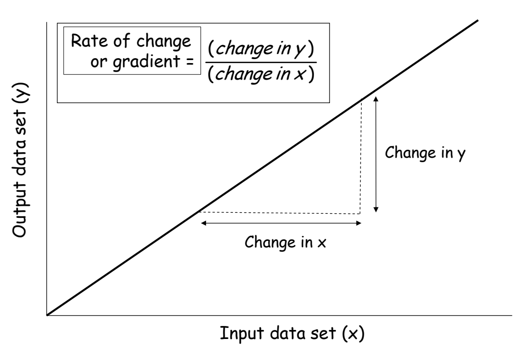
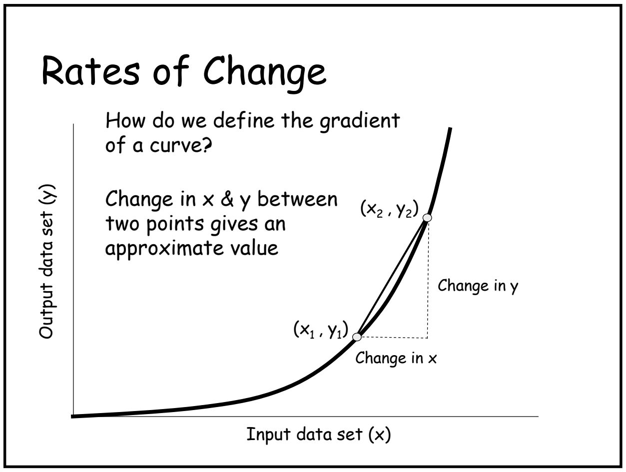
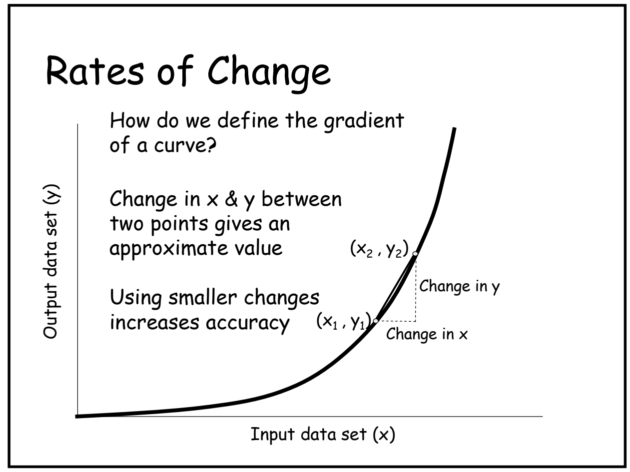
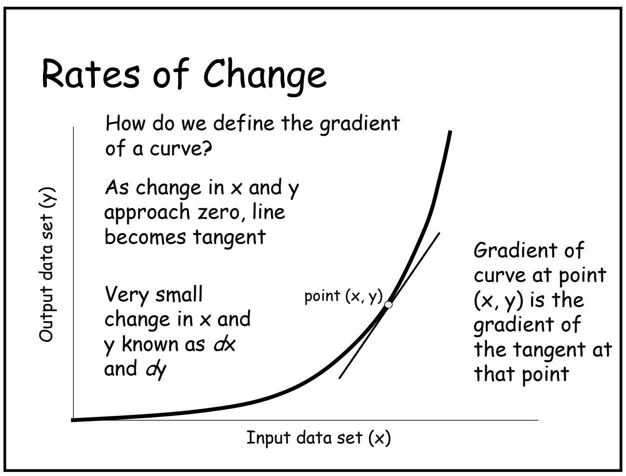


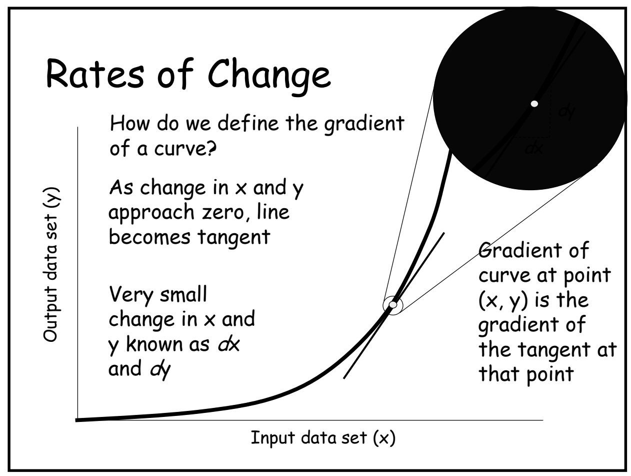
Rates of Change

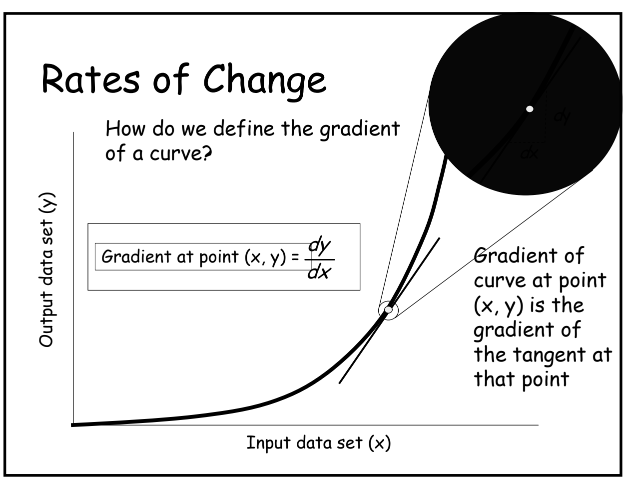












Exponentials & Logarithms

Exponentials

- $10^1=10$
- $10^2=10 \times 10 = 100$
- $10^3=10 \times 10 \times 10 = 1{,}000$
- $10^4=10 \times 10 \times 10 \times 10 = 10{,}000$
- $10^5=10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$

Exponentials

- $10^{-1}=0.1$
- $10^{-2}=10 \div 10 = 0.01$
- $10^{-3}=10 \div 10 \div 10 = 0.001$
- $10^{-4}=10 \div 10 \div 10 \div 10 = 0.0001$
- $10^{-5}=10 \div 10 \div 10 \div 10 \div 10 = 0.00001$

Logarithms

- Reverse of an exponential

$$\log_a(a^x) = x$$

Logarithms

<u>Examples</u>

$\text{Log}_{10}(100) =$

$\text{Log}_2(8) =$

$\text{Log}_3(9) =$

The number e & natural logarithms

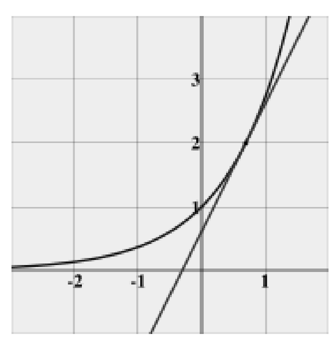
- e = 2.71828…
- Natural Log = Log_e
- Usually written as *ln*

The number e & natural logarithms

<u>Euler's number</u>

e is a unique number

The value of the slope of $f_{(x)}=e^x$ for any value of x is equal to the value of $f_{(x)}$.



Changing Base

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Changing Bases

Example:

Convert $\text{Log}_{10}(x)$ to base e

$$\log_{10}(x) = \frac{\log_e(x)}{\log_e(10)} = \frac{\ln(x)}{2.3025}$$

Exponential Functions

General Form

$$y = a^x$$

or

$$f_{(x)} = a^x$$

a is a constant and called the *base*
It can be any positive real number

Exponential Functions

Exponential Relationships

- *Arise when growth or decay of a substance is proportional to original amount of substance*
- Examples ?

Exponential Functions

Example: Exponential Growth

- A particular bacteria doubles every day.
- If the initial number of bacteria (N_o) is 100.
  - After ONE day there are 200 bacteria (N=200)
  - After TWO days there are 400 bacteria (N=400)
  - After THREE days there are 800 bacteria (N=800)
  - And so on...

- How many bacteria are there after 8 days?
- How many bacteria are there after 1000 days?
- Need to create mathematical function

Exponential Functions

Example: Exponential Growth

- A particular bacteria doubles every day.
- If the initial number of bacteria (N_o) is 100.

- After ONE day $N = N_o \text{ X } 2$
- After TWO days $N = N_o \text{ X } 2 \text{ X } 2$
- After THREE days $N = N_o \text{ X } 2 \text{ X } 2 \text{ X } 2$
- After FOUR days $N = N_o \text{ X } 2 \text{ X } 2 \text{ X } 2 \text{ X } 2$

Exponential Functions

Example: Exponential Growth

- A particular bacteria doubles every day.
- If the initial number of bacteria (N_o) is 100.

- After ONE day $N = N_o X 2^1$
- After TWO days $N = N_o X 2^2$
- After THREE days $N = N_o X 2^3$
- After FOUR days $N = N_o X 2^4$

- For n number of days $N = N_o X 2^n$

Exponential Functions

Example: Exponential Growth

- A particular bacteria doubles every day.
- If the initial number of bacteria (N_o) is 100.

- For n number of days $N = N_o X 2^n$

After 360 days x = 360
Number of bacteria,
$N = N_o X 2^x$
$N = 100 X 2^{360}$,
$N = 100 X 2.3 X 10^{108}$
$N = 2.3 X 10^{110}$

$N = 2.3 X 10^{110}$

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$N = 2.3 X 10^{110}$

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Exponential Functions

Example: Exponential Decay

- Decay of a radionuclide.
- If the initial number of atoms of the nuclide is N_o
  - After ONE half-life $N=N_o / 2$
  - After TWO half-lives $N=N_o / 4$
  - After THREE half-lives $N=N_o / 8$
  - And so on…

- How many are there after 8 half-lives?
- How many are there after 1000 half-lives?
- -> Mathematical function

Exponential Functions

Example: Exponential Decay

- Decay of a radionuclide.
- If the initial number of atoms of the nuclide is N_o

- After ONE half-life $N = N_o /2$
- After TWO half-lives $N = N_o /4$
- After THREE half-lives $N = N_o /8$
- After FOUR half-lives $N = N_o /16$

Exponential Functions

Example: Exponential Decay

- Decay of a radionuclide.
- If the initial number of atoms of the nuclide is N_o

- After ONE half-life $N = N_o/2$
- After TWO half-lives $N = N_o/4$
- After THREE half-lives $N = N_o/8$
- After FOUR half-lives $N = N_o/16$

- For n number of days $N = N_o \times 2^{-n}$

Exponential Functions

Definition of an Exponential Relationship

"A quantity *y* is said to vary exponentially with *x* if equal changes in *x* produce equal fractional changes in *y*"

I.e. fractional change in *y* is proportional to change in *x*

Exponential Functions

- The increase/decrease in *y* is often written as *dy*
- Therefore the fractional changes in *y* is *dy/y*
- Constant of proportionality, *k*
  - Otherwise known as growth/decay constant

$$\frac{dy}{y} = k \times dx \qquad y = y_0 \times e^{kx}$$

$$\frac{dy}{y} = \frac{1}{k\,dx} \qquad y = y_0 \times e^{-kx}$$

Exponential Functions

Relationship between decay constant & half value

$$y = y_0 \times e^{-kx}$$

$$\frac{y}{y_o} = e^{-kx}$$

$$\frac{1}{2} = e^{-kx_{1/2}}$$

$$\ln\left(\frac{1}{2}\right) = -kx_{1/2}$$

$$x_{1/2} = -\ln\left(\frac{1}{2}\right)/k$$

$$k = \ln(2)/x_{1/2}$$

Exponential Functions

Example: Radioactive decay

The half-life of a particular radionuclide is 8 days.
Calculate the decay constant?

Exponential Functions

Example: Radioactive decay

$$A = A_o \times 2^{-n}$$

$$A = A_o \times e^{-\lambda t}$$

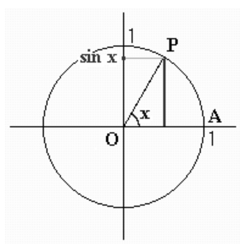
Trigonometric Functions

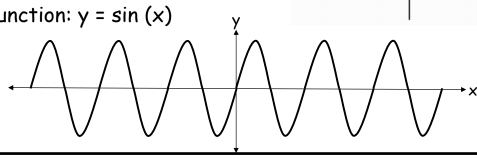
Trigonometric Functions

- Sine
  y = sin (x) or f(x) = sin (x)
- Cosine
  y = cos (x) or f(x) = cos (x)
- Tangent
  y = tan (x) or f(x) = tan (x)

Trigonometric Functions

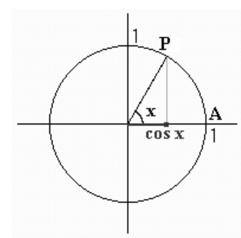
- Definition of sine function.
- The unit circle is the circle with its centre at the origin and a radius of 1.
- Angle x is formed by rotating OA about the origin to OP. Then the <u>y-coordinate</u> of point P is sin (x).
- Function: y = sin (x)

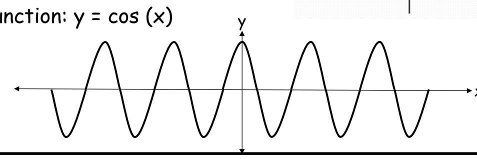




Trigonometric Functions

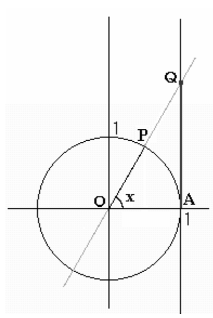
- Definition of cosine function.
- The unit circle is the circle with its center at the origin and a radius of 1.
- Angle x is formed by rotating OA about the origin to OP. Then the <u>x-coordinate</u> of point P is cos (x).
- Function: y = cos (x)

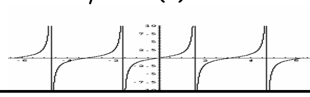




Trigonometric Functions

- Definition of tangent function.
- The unit circle is the circle with its centre at the origin and a radius of 1.
- Angle x is formed by rotating OA about the origin to OP. Point Q is the intersection of line OP and x=1.
- Then the <u>y-coordinate</u> of point Q is tan x.
- Function: y = tan (x)





Example Questions

1. Show that the decay constant is $\lambda = \ln(2)/T_{1/2}$
2. The half-life of Iodine 131 is eight days. Calculate the decay constant in (a) $days^{-1}$, (b) $seconds^{-1}$.
3. The initial activity of a radionuclide is 1MBq. What is it's half-life if after 24 hours the activity has dropped to 1,100Bq?
4. Without the use of a calculator, calculate,
   a. $\log_9(81)$
   b. $\log_7(49)$
   c. $\ln(e^{12})$